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Diffusion approximation in Comptonization process of hard x-ray passing through a 'cold' plasma—a generalized Kompaneets equation

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Abstract. The down-Comptonization scattering of hard x-ray radiation passing through an optically thick 'cold' plasma is an important radiative transfer process in x-ray astronomy as well as in radiation physics. In this paper, an extended Kompaneets equation which is valid both for up-Comptonization ($h\bar{\nu} \ll kT_e$) and down-Comptonization ($h\bar{\nu} \gg kT_e$) is derived. The numerical solutions of this resultant equation in the down-Comptonization process (i.e. in the case $h\bar{\nu} \gg kT_e$, which we are interested in) are presented, e.g. the shift of the emission-line centroid and the asymmetry of the line-profile, and the steeplization of an initial contiuum spectrum with a power-law form $I_\nu \sim \nu^{-\alpha}$. The potential applications of this improved equation in astrophysics and radiation physics are emphasized.

1. Introduction

The study of the radiative transfer process in plasma is an important topic in astrophysics, in plasma physics, and in radiation physics [1]. It is the exchange of energy between the radiation field and the plasma that causes the variations of the emergent radiation, e.g. the spectrum, the intensity, the profile of an emission line, the line shift, the intensity-ratio of different lines, and the polarization, etc. Therefore, it is necessary to give a detailed consideration of the problem of the energy transfer in order to explain some experimental laboratory results, as well as the astronomical observations and to understand the physics of some astronomical objects. We emphasize the latter because our main interest is in this field.

For a fully ionized plasma, the radiative transfer has a particular property, where the dominant mechanism of exchanging energy between the radiation and plasma is photon–electron scattering. The change of the emergent radiation and the accompanying variation of temperature of the electron gas in this process is known as Comptonization. Although the change of the photon's wavelength in each individual collision is very small, the integrated variation of frequency in multiple Compton scattering, which often occurs in the astronomical case, is remarkable. Therefore the scattering (Comptonization) is a very efficient mechanism to change the emergent spectrum, particularly for the x-ray spectrum. The efficiency of the Comptonization process can be evaluated as follows. It is well known that the change of photon wavelength in each collision is $\Delta\lambda/\lambda = (2\lambda_c/\lambda)\sin^2\theta/2 \simeq \lambda_c/\lambda$, where $\lambda_c = 0.024$ Å is the Compton wavelength and θ is the scattering angle. Therefore the fractional change $\Delta\lambda/\lambda$ is dependent on the initial wavelength. The shorter the initial λ, the larger the change $\Delta\lambda/\lambda$. For example, the fractional shift is $\Delta\lambda/\lambda \simeq 10^{-6}$ if $\lambda = 5000$ Å but $\Delta\lambda/\lambda \simeq 10^{-2}$ if $\lambda = 0.5$ Å ($h\nu = 20$ keV); as for the very hard x-ray photon with energy ~ 100 keV, $\Delta\lambda/\lambda \simeq 10^{-1}$. That is why it is particularly important for x-ray and γ-ray radiation.

In principle, a strict theoretical approach to the Comptonization should be based on the equation of radiative transfer. However, for the photon–electron scattering process, this equation turns out to be rather complicated. The emissivity j_ν in this equation has to be expressed by an integral of the intensity I_ν at the same space-point. Therefore, we have a complicated integral-differential equation which is difficult to solve. Another approach to photon–electron scattering is the Monte Carlo simulation, but the numerical calculation for this problem is troublesome. In order to avoid such difficulties, Kompaneets developed a more convenient method—the diffusion approximation. The basic idea is that the whole studied system is regarded as a mixed gas which consists of a fully ionized plasma and a radiation field, where the change of the frequency spectrum of the radiation field, due to the Compton scattering, is formally considered as a 'diffusion process' of a photon gas in 'frequency space'. In this approach, the photon's frequency is approximated as a continuous variable because of the very small change of the photon's frequency in each collision, i.e. $\Delta \equiv \nu' - \nu, |\Delta| \ll \nu$. When the mixed gas reaches thermal equilibrium the diffusion process ceases, and the frequency spectrum turns out to be invariable. In order to describe quantitatively the variation of the spectrum with time before reaching thermal equilibrium, it is necessary to establish a dynamical diffusion equation for the distribution function of photon frequency $n(\nu, t)$.

There are two kinds of Comptonizations which have different diffusion equations. If the average energy of photons, $h\bar{\nu}$, is much larger than the thermal energy of an electron kT_e, $h\bar{\nu} \gg kT_e$, which often occurs in x-ray and γ-ray astronomy, then the average energy of scattered photons will decrease. This is known as down-Comptonization or Comptonization-softening. On the other hand, the energy of scattered photons will increase when $h\bar{\nu} \ll kT_e$, which is known as up-Comptonization or Comptonization-hardening, which often occurs in radio and infrared astronomy. Under the condition $h\bar{\nu} \ll kT_e \ll M_e c^2$ (up-Comptonization), the diffusion equation is known as Kompaneets equation [2]:

$$\frac{\partial n}{\partial t} = \frac{kT_e}{M_e c^2} N_e \sigma_T c \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ x^4 \left[\frac{\partial n}{\partial x} + n(n+1) \right] \right\} \tag{1}$$

where $x \equiv h\nu/kT_e$ is a dimensionless photon frequency, σ_T is the Thomson cross section, N_e is the number density of electron gas, and $n(x, t) = n(\nu, t)$ is the frequency distribution function of the photon gas, which represents the 'photon's number' in each photon state in unit volume at frequency ν. Therefore, the real number density of photons in $\nu \to \nu + d\nu$ will be $n(\nu, t)(8\pi\nu^2/c^3)\,d\nu$. For a weak radiation field, the 'photon's number' $n(\nu, t) \ll 1$.

Weymann obtained the same result as that given by Kompaneets [3]. But we would like to mention that the Kompaneets equation (1) is correct only if two conditions have been satisfied, that is: (i) the system studied (a mixture of radiation field and thermal electron gas) has to be a non-relativistic one, i.e. $kT_e \ll M_e c^2$ and $h\nu \ll M_e c^2$, and (ii) the system studied must be one which consists of a thermal electron gas with high temperature and a radiation field with low frequency, i.e. $h\nu \ll kT_e$. Therefore the application of Kompaneets equation (1) is rather limited.

After this, Cooper extended the Kompaneets equation (1) from the non-relativistic to the relativistic system [4]:

$$q^2 \frac{\partial n(q,t)}{\partial t} = \frac{\partial}{\partial q}\left\{\alpha(q,kT_e)\left[n(q,t)(1+n(q,t)) + kT_e \frac{\partial n(q,t)}{\partial q}\right]\right\} \tag{2}$$

where $q \equiv h\nu, \alpha(q,kT_e) = \alpha_{NR}[1+f(kT_e)/(1+0.02q)][(1+9\times10^{-3}q+4.2\times10^{-6}q^2)]^{-1}$.

Cooper's equation (2) can be used to discuss a scattering gas with very high temperature. But we noticed that the condition (ii), $h\bar{\nu} \ll kT_e$, is still necessary for Cooper's equation. That is, the studied system must be one with high temperature and low frequency.

Differing from Kompaneets, Weymann and Cooper, we emphasize the importance of a non-relativistic scattering process with condition $h\bar{\nu} \gg kT_e$ in practice (say, a hard x-ray passing through a 'cold' plasma). We call such a process down-Comptonization, which is potentialy important in hard x-ray astronomy and physics [5–7]. But, so far, there is no appropriate quantitative description for the down-Comptonization in the case $h\bar{\nu} \gg kT_e$. In this paper, an extened Kompaneets equation which is valid both for up-Comptonization ($h\bar{\nu} < kT_e$) and down-Comptonization ($h\bar{\nu} > kT_e$), including $h\bar{\nu} \sim kT_e$, is given and its basic properties and potential applications in x-ray astronomy are discussed in section 2. In section 3, some conclusions and discussions are given. Some instructive solutions of this equation for typical astronomical problems are presented in section 4.

2. The extended Kompaneets equation for $kT_e \ll M_e c^2$ and $h\nu \ll M_e c^2$

Consider a 'mixed gas' consisting of a plasma and a photon gas under the condition $kT_e \ll M_e c^2$ and $h\nu \ll M_e c^2$. We make two assumptions. (i) The common thermal equilibrium between these two gases is not yet reached, but the electron gas itself is already in thermal equilibrium due to the fact that the interaction between electrons is the Coulomb long-range force. Therefore, the Maxwellian distribution $f(\boldsymbol{p}) = f_0 \exp(-p^2/2M_e kT_e)$ can be used to describe the electron gas. (ii) Because of the frequent Compton scattering, the distribution of photon-frequency $n(\nu,t)$ is assumed to be isotropic, independent of the directions of the wavevector $\boldsymbol{k}$ and $n(\nu,t)$ will change with time before reaching the thermal equilibrium. For such a mixed-gas system, the change rate of the 'photon's number' $\partial n/\partial t$ is given by the diffusion equation which is derived as follows.

Consider an individual collision between an electron with momentum $\boldsymbol{p}$ and a photon with frequency ν, the energy and momentum conservations in the non-relativistic limit are

$$\left(\frac{h\nu}{c}\right)\boldsymbol{n} + \boldsymbol{p} = \left(\frac{h\nu'}{c}\right)\boldsymbol{n}' + \boldsymbol{p}' \qquad h\nu + \frac{p^2}{2M_e} = h\nu' + \frac{p'^2}{2M_e} \tag{3}$$

where $\boldsymbol{p}'$ and ν' represent the momentum of the electron and the frequency of photon, respectively, after the collision, $\boldsymbol{n}$ and $\boldsymbol{n}'$ are the directions of the photon before and after the collision. Such a process $(\boldsymbol{p}, \nu, \boldsymbol{n}) \longrightarrow (\boldsymbol{p}', \boldsymbol{n}')$ leads to a decrease of the photon number $n(\nu,t)$. Denoting the transition probability of this process by $\mathrm{d}W$ and the electron density in $\boldsymbol{p} \longrightarrow \boldsymbol{p} + \mathrm{d}\boldsymbol{p}$ by $N_e f(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}$, taking the tenuous electron gas as a classical system, and the photon gas as a boson system, the transition number of this collision is

$$(1+n')nN_e f(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}W$$

where $n \equiv n(\nu,t), n' \equiv n(\nu',t)$ are the photon number before and after the collision, respectively. Similarly, the inverse process $(\boldsymbol{p}', \nu', \boldsymbol{n}') \longrightarrow (\boldsymbol{p}, \nu, \boldsymbol{n})$ leads to an increase of $n(\nu,t)$. The transition number is

$$(1+n)n' N_e f(\boldsymbol{p}')\,\mathrm{d}^3\boldsymbol{p}'\,\mathrm{d}W\,.$$

Therefore, the change-rate $\partial n/\partial t$ can be written as

$$\frac{\partial n}{\partial t} = -N_e \int d^3p \int dW[n(1+n')f(p) - n'(1+n)f(p')] . \tag{4}$$

Here we would like to mention that the transition probability dW is the same in the two processes $(\boldsymbol{p}, \nu, \boldsymbol{n}) \rightleftharpoons (\boldsymbol{p}', \nu', \boldsymbol{n}')$, the reason is that we can approximately replace the Klein–Nishina cross section by the classical Thomson cross section $d\sigma_T/d\theta$ which has the same value for both the scattering angle θ and $\pi - \theta$ (see (10)). It can be seen that it should be zero if we insert the thermal equilibrium distribution $n(\nu) = (e^{h\nu/kT_e} - 1)^{-1}$ and $f(p) = f_0 \exp(-p^2/2M_e kT_e)$ into (4), indicating that the Comptonization is effective only before reaching the common thermal equilibrium.

Equation (4) can be simplified since the electron gas is non-relativistic, i.e. $kT_e \ll M_e c^2$, so the change of frequency is very small in each collision, $\Delta \equiv \nu' - \nu$, $|\Delta| \ll \nu$. Expanding in terms of the small quantity Δ to second order under the condition $h\nu|\Delta| \ll kT_e$, and replacing the frequency ν by a dimensionless frequency x, $x \equiv h\nu/kT_e$, equation (4) becomes

$$\frac{\partial n}{\partial t} = \left[\frac{\partial n}{\partial x} + n(n+1)\right]\frac{N_e h}{kT_e}\int d^3p \int dW\, f(p)\Delta + \left[\frac{\partial^2 n}{\partial x^2} + 2(n+1)\frac{\partial n}{\partial x} + n(n+1)\right]\frac{N_e}{2}\left(\frac{h}{kT_e}\right)^2 \int d^3p \int dW\, f(p)\Delta^2 . \tag{5}$$

In the following derivations, we first calculate the second integral on the right-hand side of (5). Let

$$I \equiv h^2 \int d^3p \int dW\, f(p)\Delta^2 . \tag{6}$$

The expression $\Delta \equiv \nu' - \nu$ in the integral can be obtained from (3), and only retaining the first order of the small quantity Δ, we obtain

$$h\Delta = -\frac{hc\nu \boldsymbol{p}\cdot(\boldsymbol{n} - \boldsymbol{n}') + (h\nu)^2(1 - \boldsymbol{n}\cdot\boldsymbol{n}')}{M_e c^2[1 + (h\nu/M_e c^2)(1 - \boldsymbol{n}\cdot\boldsymbol{n}') - (\boldsymbol{p}\cdot\boldsymbol{n}'/M_e c^2)]} . \tag{7}$$

Because of $h\nu \ll M_e c^2$ and $kT_e \ll M_e c^2$, the value in the square bracket in the denominator of (7) nearly equals 1. Thus, we have

$$h\Delta = -\frac{h\nu c}{M_e c^2}\boldsymbol{p}\cdot(\boldsymbol{n} - \boldsymbol{n}') - \frac{(h\nu)^2}{M_e c^2}(1 - \boldsymbol{n}\cdot\boldsymbol{n}') . \tag{8}$$

Under the condition $kT_e \ll h\nu \ll M_e c^2$, the second term in (8) will be comparable with the first term and cannot be neglected. Inserting (8) into (7), the integral I can be reduced to

$$\begin{aligned} I &= h^2 \int d^3p \int dW\, f(p)\Delta \\ &= \int d^3p \int dW\, f(p)\left[-\frac{hc\nu}{M_e c^2}\boldsymbol{p}\cdot(\boldsymbol{n} - \boldsymbol{n}') - \frac{(h\nu)^2}{M_e c^2}(1 - \boldsymbol{n}\cdot\boldsymbol{n}')\right]^2 \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{9}$$

with

$$I_1 = \left(\frac{h\nu}{M_e c}\right)^2 \int d^3\boldsymbol{p} \int dW\, f(\boldsymbol{p}) |\boldsymbol{p}\cdot(\boldsymbol{n}-\boldsymbol{n}')|^2$$

$$I_2 = \left(\frac{h^2\nu^2}{M_e c^2}\right)^2 \int d^3\boldsymbol{p} \int dW\, f(\boldsymbol{p})(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2$$

$$I_3 = 2\frac{(h\nu)^3}{M_e^2 c^3} \int d^3\boldsymbol{p} \int dW\, f(\boldsymbol{p})\boldsymbol{p}\cdot(\boldsymbol{n}-\boldsymbol{n}')(1-\boldsymbol{n}\cdot\boldsymbol{n}')\,.$$

Fixing $\boldsymbol{n}-\boldsymbol{n}'$ as the z-axis and denoting the angle between $\boldsymbol{p}$ and $\boldsymbol{n}-\boldsymbol{n}'$ as Θ, we have

$$I_1 = \left(\frac{h\nu}{M_e c}\right)^2 \int d^3\boldsymbol{p} \int dW\, p^2|\boldsymbol{n}-\boldsymbol{n}'|\cos^2\Theta f(\boldsymbol{p})$$

$$= \left[\left(\frac{h\nu}{M_e c}\right)^2 \int dW|\boldsymbol{n}-\boldsymbol{n}'|^2\right] \int p^4 f(p)\cos^2\Theta\,\sin\Theta\, dp\, d\Theta\, d\varphi$$

$$= \frac{1}{3}\int p^2 f(p) 4\pi p^2\, dp \left[\left(\frac{h\nu}{M_e c}\right)^2 \int dW|\boldsymbol{n}-\boldsymbol{n}'|^2\right].$$

Inserting $f(p) = f_0 \exp(-p^2/2M_e kT_e)$ into the expression I_1, we have $\int p^2 f(p) 4\pi p^2\, dp = 3M_e kT_e$, therefore

$$I_1 = \left(\frac{h\nu}{M_e c}\right)^2 M_e kT_e \int dW|\boldsymbol{n}-\boldsymbol{n}'|^2 = \left(\frac{h\nu}{M_e c}\right)^2 2M_e kT_e \int dW(1-\boldsymbol{n}\cdot\boldsymbol{n}')\,.$$

Similarly we have

$$I_2 = \left(\frac{h^2\nu^2}{M_e c^2}\right)^2 \int d^3\boldsymbol{p}\, f(\boldsymbol{p}) \int dW(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2 = \left(\frac{h^2\nu^2}{M_e c^2}\right)^2 \int dW(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2$$

$$I_3 = \frac{2(h\nu)^3}{M_e^2 c^3}\left[\int dW|\boldsymbol{n}-\boldsymbol{n}'|(1-\boldsymbol{n}\cdot\boldsymbol{n}')\right] \int p^3 f(p)\cos\Theta\sin\Theta\, dp\, d\Theta\, d\varphi = 0\,.$$

Therefore, we have to calculate the integrals $\int dW(1-\boldsymbol{n}\cdot\boldsymbol{n}')$ and $\int dW(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2$ for all scattering directions θ. In the non-relativistic limit, the Compton differential scattering cross section can be approximately replaced by the Thomson cross section, so the transition probability is

$$dW = c\, d\sigma_T = c\frac{r_0^2}{2}(1+\cos^2\theta)2\pi\sin\theta\, d\theta \tag{10}$$

where $r_0 = e^2/M_e c^2$ is the classical radius of the electron, θ is the scattering angle, $\cos\theta = \boldsymbol{n}\cdot\boldsymbol{n}'$. From (10) we see that dW is symmetric for angles θ and $\pi-\theta$. Therefore, $\int \boldsymbol{n}\cdot\boldsymbol{n}'\, dW = \int\cos\theta\, dW = 0$ and $\int(1-\boldsymbol{n}\cdot\boldsymbol{n}')\, dW = \int dW = \sigma_T c$, $\int(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2\, dW = \pi c r_0^2 \int(1+\cos^2\theta)^2\sin\theta\, d\theta = 7/5\sigma_T c$. Hence,

$$I_1 = \frac{2kT_e}{M_e c^2}(h\nu)^2\sigma_T c \qquad I_2 = \frac{(h\nu)^4}{M_e{}^2 c^4}\frac{7}{5}\sigma_T c$$

and

$$I = I_1 + I_2\,. \tag{11}$$

Now we calculate the first integral in (7). Let

$$H \equiv h\int d^3\boldsymbol{p}\int dW\, f(\boldsymbol{p})\Delta\,. \tag{12}$$

It is difficult to calculate (12) directly, but it can be deduced simply from the integral value I given by (11) from consideration of the conservation of the total number of photons in the whole photon–electron scattering process. The conservation equation in the 'frequency space' can be written as [8]

$$\frac{\partial n}{\partial t} = -\nabla \cdot j = -x^{-2}\frac{\partial(x^2 j)}{\partial x} \qquad (13)$$

where j is the flux of photons defined in the frequency space where the Cartesian coordinates are x_1, x_2, x_3 respectively, therefore $\sqrt{x_1 + x_2 + x_3} = x = h\nu/kT_e$. Using the spherical (x, θ, φ) to replace (x_1, x_2, x_3), we have $\nabla \cdot j = (1/x_2)\partial(x^2 j)/\partial x$ and (13) is obtained. Equation (13) can be rewritten as

$$\frac{\partial n}{\partial t} = -\frac{2}{x}j - \frac{\partial j}{\partial x}. \qquad (14)$$

Comparing (14) with (5), it is seen that the flux of photons j has to be taken in the form

$$j(x) = g(x)\left[\frac{\partial n}{\partial x} + n(n+1)\right]. \qquad (15)$$

The reason is that in (5) $\partial n/\partial t$ is linearly dependent on the second derivative $\partial^2 n/\partial x^2$, i.e. the coefficient of $\partial^2 n/\partial x^2$ in (5) does not contain n. Therefore it is seen from (14) that only the form $j(x) \sim [\partial n/\partial x + f(n)]$ or $j(x) = g(x)[\partial n/\partial x + f(n)]$ is possible and the proportional coefficient $g(x)$ is independent of n. Furthermore, it can be deduced that the function $f(n)$ must be taken as $f(n) = n(n+1)$ due to the fact that when the photon gas reaches thermal equilibrium we have $j(x) = 0$, therefore $\partial n/\partial x = -f(n)$. On the other hand, in this case the distribution function is Planckian, $n(x) = (e^x - 1)^{-1}$, therefore $\partial n/\partial x = -n(n+1)$. So, we get $f(n) = n(n+1)$, and (15) is obtained where $g(x)$ is a coefficient which is to be determined. Inserting (15) into (14), we get

$$\frac{\partial n}{\partial t} = -g(x)\left[\frac{\partial^2 n}{\partial x^2} + (2n+1)\frac{\partial n}{\partial x}\right] - \left[\frac{\partial g}{\partial x} + \frac{2g}{x}\right]\left[\frac{\partial n}{\partial x} + n(n+1)\right]. \qquad (16)$$

On the other hand, equation (5) can be written as

$$\frac{\partial n}{\partial t} = \left[\frac{\partial n}{\partial x} + n(n+1)\right]\frac{N_e}{kT_e}H + \frac{N_e}{2}(kT_e)^{-2}I\left[\frac{\partial^2 n}{\partial x^2} + 2(n+1)\frac{\partial n}{\partial x} + n(n+1)\right]. \qquad (17)$$

Comparing (16) with (17) and noting that the coefficient of $\partial^2 n/\partial x^2$ should be the same, $g(x)$ is obtained as

$$g(x) = -\frac{N_e}{2}(kT_e)^{-2}I = -Ax^2(1 + Bx^2) \qquad (18)$$

where $A(x) \equiv (kT_e/M_e c^2)N_e\sigma_T c$, $B(x) \equiv \frac{7}{10}(kT_e/M_e c^2)$. Inserting (18) into (16) and comparing (16) with (17) again, we obtain

$$H = \frac{kT_e}{N_e}A[4x - x^2 + 6x^3 - Bx^4]. \qquad (19)$$

Therefore, the extended Kompaneets equation under the condition $kT_e \ll M_e c^2$, and $h\nu \ll M_e c^2$ is obtained as follows:

$$\frac{\partial n}{\partial t} = \frac{kT_e}{M_e c^2}N_e\sigma_T c\frac{1}{x^2}\frac{\partial}{\partial x}\left\{x^4\left(1 + \frac{7}{10}\frac{kT_e}{M_e c^2}x^2\right)\left[\frac{\partial n}{\partial x} + n(n+1)\right]\right\}. \qquad (20)$$

3. Conclusion and discussions

The extended Kompaneets equation (20) is useful in astrophysics and radiation physics. In particular, the condition $kT_e \ll h\nu \ll M_e c^2$ is usual in laboratories and in astrophysical circumstances. For example, the condition $kT_e \ll h\nu < M_e c^2$ is satisfied when the hard x-ray radiation with average energy of photons $h\nu$ = 10–100 keV is passing through a thermal plasma with $T_e = 10^7$ K. Obviously, in this case, the emergent spectrum will be changed. Usually there is only a qualitative or semi-quantitative approach to the down-Comptonization process because of the fact that the Monte Carlo calculation is too complicated and is not very helpful for insight into the physics of the process [9, 10]. The diffusion equation (20) presented in this paper is appropriate to quantitatively describe the down-Comptonization, which can be used to replace the usual radiative transfer equation and some potential applications would be expected (see section 4).

Here we would like to point out some properties of equation (20). (i) The structure $(1/x^2)\partial/\partial x\{\ldots\}$ ensures the conservation of the number of photons in the scattering process (cf (20) with the continuity equation (13)). (ii) The structure $\partial n/\partial x + n(n+1)$ ensures $\partial n/\partial t = 0$ when the radiation field reaches thermal equilibrium. (iii) In the low-frequency limit, i.e. if $x \equiv h\nu/kT_e \ll 1$, equation (20) returns to the Kompaneets equation (1). (iv) Equation (20) is only suitable to describe the 'diffusion' in the 'frequency space'. If the radiation field itself is also inhomogeneous in space, then $n = n(x, \boldsymbol{r}, t)$, and a space-diffusion term $\nabla \cdot (D\nabla n)$ must be added to (20). Then (20) becomes

$$\frac{\partial n}{\partial t} = \frac{kT_e}{M_e c^2} N_e \sigma_T c \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ x^4 \left(1 + \frac{7}{10} \frac{kT_e}{M_e c^2} x^2 \right) \left[\frac{\partial n}{\partial x} + n(n+1) \right] \right\} + \nabla \cdot (D\nabla n) . \qquad (21)$$

4. Some typical numerical solutions

Equation (21) has potential applications in x-ray and γ-ray astronomy [11, 12]. As an example, we discuss an emergent spectrum of an x-ray point source (e.g. in an x-ray binary). For the Comptonization of photons in a finite medium, the isothermal uniform-sphere model is generally accepted [3], where an x-ray point source is surrounded by an isothermal sphere of cooler gas. One can imagine the x-ray point source as a compact star, e.g. a neutron star, white dwarf or black hole, and the cooler plasma as the accretion gas which surrounds the star. The x-ray emitted from a central point source is subjected to the down-Comptonization in this sphere. Ross *et al* pointed out that, for the compact x-ray source, this model is close to reality, althougth such an idealized model is still bound to be a crude approximation to the very complex geometric and thermal structure [5].

In order to study the effect of down-Comptonization on the emergent x-ray spectrum, equation (21) must be solved numerically with given initial and boundary conditions. In spherical coordinates, in principle, the initial and boundary conditions of (21) can be taken as follows:

$$\begin{aligned} & n(x, 0, t) = f(x) \qquad t \geqslant 0 \\ & n(x, r, 0) = 0 \qquad 0 < r < R \\ & -D \left.\frac{\partial n}{\partial r}\right|_{r=R} = \frac{c}{2} n(x, R, t) . \end{aligned} \qquad (22)$$

The first two conditions mean that, from $t = 0$, there is a stable x-ray source in the centre of the sphere and the frequency distribution function of photons in the source is $f(x)$. The boundary condition (see the last condition) is called Eddington's approximation which is suitable under the assumption of isotropic radiation, where $I(x,t) = -D\partial n/\partial r|_{r=R}$ is the flux density of photons on the surface of the sphere.

Solving (21) under conditions (22), the frequency distribution function $n(x, r, t)$ can be obtained and the emergent x-ray spectrum should be

$$I(x,t) \sim x^3 n(x, R, t). \tag{23}$$

Because there exists a central stable source, when t is large enough, $n(x, r, t)$ will be unchanged with t, and

$$F(x) \sim x^3 n(x, R) \tag{24}$$

where $F(x)$ is the required x-ray spectrum after down-Comptonization. If a pulse flux of x-rays with a frequency distribution function $f(x)$ is emitted at the centre of the sphere at $t = 0$, the first condition in (22) is replaced by

$$n(x, 0, t) = f(x)\delta(t). \tag{25}$$

The x-ray spectrum can be written in the form

$$F(x) \sim x^3 \int_0^\infty n(x, R, t)\,\mathrm{d}t. \tag{26}$$

Equations (24) and (26) are equivalent to each other. Equation (24) represents the emergent spectrum of photons from the surface of the sphere at any given time t but which are emitted from the centre of the sphere at different times. However, equation (26) is a spectrum of photons which are emitted from the centre at an instant of time, but which emerge from the surface at different times.

Equation (21) can be further simplified by separation of variables. For the compact x-ray source in normal conditions, the radiation field is often thin enough, i.e. $n(x, r, t) \ll 1$. If we neglect the term with order n^2, equation (21) becomes a linear-differential equation. Furthermore, the diffusion coefficient in (21) is $D = c/3N_e\sigma_c \simeq c/3N_e\sigma_T$ = constant due to the fact that the Klein–Nishina cross section σ_c is slowly varying with frequency ν and $\sigma_c = \sigma_T$ (Thomson cross section) in the x-ray frequency range. Putting $n(x, r, t) \equiv W(x, t)P(r, t)$, in spherical coodinates, equation (21) and its condition (22) can be written in two sets of independent forms:

$$\begin{aligned} &\frac{\partial W}{\partial t} + \frac{kT_e}{M_e c^2} N_e \sigma_T c \frac{1}{x^2}\frac{\partial}{\partial x}\left\{x^4\left(1 + \frac{7}{10}\frac{kT_e}{M_e c^2}x^2\right)\left[\frac{\partial W}{\partial x} + W\right]\right\} = 0 \\ &W(x, 0) = f(x) \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\frac{\partial P}{\partial t} + D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = 0 \qquad P(0,0) = 1 \quad p(r,0) = 0 \quad 0 < r < R \\ &-D\frac{\partial P}{\partial r}\bigg|_{r=R} = \frac{c}{2}P(R, t). \end{aligned} \tag{28}$$

Because the emission and absorption of the scattering medium itself is neglected, this diffusion equation does not contain a source, so we put zero on the right-hand side of (27) and (28). Equation (28) is a standard space-diffusion equation and $P(r,t)|_R$, which is independent of frequency x, is obviously related to the escape probability of x-ray photons from the surface $r = R$. $W(x, t)$ represents a distribution of photons in frequency space, which is varied with time during the down-Comptonization process.

In the following, only (27) is used to discuss the effect of down-Comptonization on the emergent spectrum because $P(R, t)$ is an unimportant factor for the whole x-ray spectrum (see (26)). In this paper, two typical primary spectra are considered.

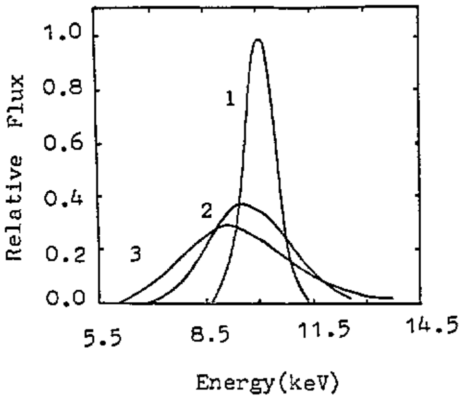

Figure 1. The evolution of an initial Gaussian spectrum. The labels 1, 2 and 3 represent times $t = 0$, 0.05 and 0.1 s, respectively.

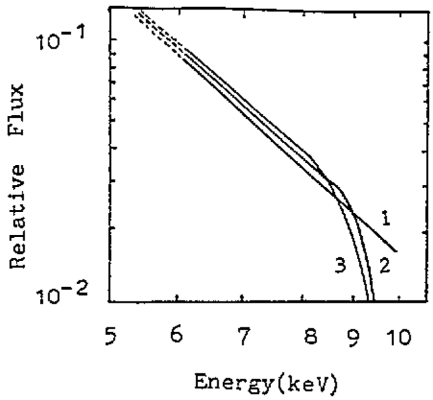

Figure 2. The evolution of an initial power-law continuum. The labels 1, 2 and 3 represent times $t = 0$, 0.025 and 0.05 s, respectively.

(a) An emission line with Gaussian form.

The x-ray emission line arises from H-like or He-like Fe-ions (i.e. Fe XXVI) and is often broadened by very complicated bulk motion in the source. The spectral profile usually has a Gaussian form:

$$I_\nu = A_0 \exp[-C_0(\nu - \nu_0)^2] \tag{29}$$

where ν_0 is the central frequency, both A_0 and C_0 are constants. In this case, the initial condition can be written as

$$n(x, 0) = f(x) \sim x^{-3} \exp[-C_1(x - x_0)^2] . \tag{30}$$

(b) A power-law continuum emission.

The x-ray radiation from some astronomical objects (e.g. BL Lacertae objects, Seyfert galaxies and quasars) are usually characterized by a power-law continuum

$$I_\nu = B_0 \nu^{-\alpha} \qquad \nu_1 \leqslant \nu \leqslant \nu_2 \tag{31}$$

where the energy index $\alpha \sim 0.5$–1.6 and B_0 is a constant. In this case, the initial condition is taken as

$$n(x, 0) = f(x) \sim \begin{cases} x^{-3}x^{-\alpha} & (x_1 \leqslant x \leqslant x_2) \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

where x_1, x_2 are the upper and lower limits of the energy range, respectively.

Equation (27) is solved numerically with these initial conditions by the finite-difference method, using the following parameters: $\tau_e = 5, R \sim 10^9$ cm, $N_e \sim 10^{16}$ cm^{-3} and $T_e \sim 10^7 K$ [12, 13] for a typical point source. The interesting results about the evolution of the x-ray spectrum due to down-Comptonization has been obtained. Figure 1 displays the evolution of the profile of an x-ray emission line with initial Gaussian form. It is easily seen from figure 1 that: (i) the peak intensity decreases with an increase of diffusion time; (ii) the peak position is obviously shifted downwards to lower energy, indicating that the average effect of photon–electron scattering makes photons lose their energy gradually when the x-ray photons are passing through a cooler electron gas; (iii) an asymmetric line profile appears. The red side becomes clearly steeper than the blue side, which is the result of competition between 'diffusion' ($\sim \partial^2 n/\partial x^2$ term) and 'convection' ($\sim \partial n/\partial x$ term) in

(27); (iv) the FWHM (full width of half maximum) becomes larger with increasing time t. In figure 2, the evolution of a power-law continuum (1–10 keV) with index $\alpha = 0.7$ is presented. As expected, the initial spectrum shown as an inclined line with a slope α in the $\ln I$–$\ln h\nu$ diagram is changed in such a way that the intensities in the high-energy part become lower but become higher at the low-energy end, implying that photons with high frequency have a tendency to shift to lower frequency during the down-Comptonization process, which leads to an 'accumulation' of photons with lower frequency. All of these evolution characteristics are in accordance with our intuition. Equation (21) with these two typical initial conditions is expected to be useful in interpreting the observations and understanding the physics in some astronomical objects.

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